Session 0: Getting Familiar with Fields, Symmetries and Gauges

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Abstract

First draft of the notes for session 0. Any feedback is appreciated as well as reporting typos. Good references for the material presented here can be found in [1–5].

Contents

The goal of this session is to get introduced to basic notions that will be used extensively in gauge theory. By this I hope to establish some common ground so that we can start building from there. While the material of this session should be familiar to the readers, as it is mainly definitions and notation seen in undergraduate physics courses, it will be useful to revisit it as conventions might be different to the ones typically used.

1 Crash Course in Mechanics

Let us consider an N-body system. These bodies live in a $d-$dimensional space Σ — i.e. a manifold[1] — called the *configuration space*. Each of the N bodies has a state in this space given by parameters $\{q_i^a\}$ known as the *generalised coordinates*, where $1 \leq i \leq d$ labels each of the components and $1 \leq a \leq N$ labels the body. We can define a *generalised velocity* in this configuration space by taking a time derivative of the general coordinates

$$\dot{q}_i^a = \frac{dq_i^a}{dt} \, . \tag{1}$$

We might call the quantity $q_i^a(t)$ the *trajectory* of body a in configuration space, where $t \in [t_0, t_f]$.

In this general setting we might introduce a function called the Lagrangian $L[\mathbf{q}^a, \dot{\mathbf{q}}^a] = T - V$ which is, in general, defined as the difference between the kinetic energy (T) and the potential energy (V). This allows us to define[2] a functional called the *action*, namely

$$S[q(t), \dot{q}(t)] = \int_{t_0}^{t_f} dt \; L(q, \dot{q}) \, , \tag{2}$$

as the integral over time of said Lagrangian with well-defined endpoints $q_0 = q(t_0)$ and $q_f = q(t_f)$. In short, the action takes as input a function of the generalised coordinates and velocities, sums over configuration space and returns a number as output. The action is a fundamental object in physics as it gives full control over the properties of the system under study.

> **Principle of least action (or Hamilton's principle)**
>
> The physical trajectory corresponds to an extremum of the action, meaning
>
> $$\frac{\delta S[q, \dot{q}]}{\delta q} = \frac{\partial L(q, \dot{q})}{\partial q} - \frac{d}{dt} \frac{\partial L(q, \dot{q})}{\partial \dot{q}} = 0 \, , \tag{3}$$
>
> which yields the Euler-Lagrange equation of motion in configuration space.

One can see the Euler-Lagrange equation arising by considering a variation over the action

$$\delta S = \int_{t_0}^{t_f} dt \; L\big(q + \delta q, \dot{q} + \delta \dot{q}\big) - \int_{t_0}^{t_f} dt \; L\big(q, \dot{q}\big) = \int_{t_0}^{t_f} dt \, \delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) = 0 \, , \tag{4}$$

where the boundaries are fixed, meaning $\delta q_0 = \delta q_f = 0$. We then define the *canonically conjugate momentum* (to q) as

$$p \equiv \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \, , \tag{5}$$

which allows us to re-express the Euler-Lagrange equation as

$$\dot{p} = \frac{dp}{dt} = \frac{\partial L}{\partial q} \tag{6}$$

[1] A manifold is nothing but the generalisation of the familiar notion of curves and surfaces to arbitrary dimensional objects. These generalised objects must satisfy the condition in that, at every point, they locally look "flat" (or Euclidean). Hence, a $d-$dimensional manifold (or $d-$manifold for short) locally resembles $\simeq \mathbb{R}^d$.

[2] In the following, for ease of notation, we suppress indices and introduce the action for a single particle and a single generalised coordinate. This can be extended to the more general case by restoring labels.

Newton from Euler-Lagrange

One can see that the above prescription reduces to Newton's equation of motion in the appropriate limit. For instance, should we consider a single test particle in a $d-$dimensional configuration space with Lagrangian $L(q_i, \dot{q}_i) = \frac{1}{2}m\dot{q}_i^2 - V(q_i)$. We find that the Euler-Lagrange equations of motion (one for each generalised coordinate) are

$$m\ddot{q}_i + \frac{\partial V}{\partial q_i} = 0\,, \tag{7}$$

which we recognise as Newton's equation of motion.

A particularly interesting instance of the previous general result is the case in which $q(t)$ does not explicitly appear in the Lagrangian, so $L = L(\dot{q})$. The coordinate is then said to be *cyclic*, and it immediately implies that $\dot{p} = 0$. Therefore, the conjugate momentum is constant in time, it is *conserved*. This is nothing but a particularly simple case of the more general notion of *symmetry* and as understood from *Noether's theorem*, linking the presence of symmetries with conservation of physical quantities. We will see more on this in the future.

We are almost ready to take a break and observe the view from the top of the mountain. All we need to do is define a new object, the — much beloved by physicists — Hamiltonian $H(q,p)$, which does not live in configuration space $(q,\dot{q})$, but in *phase space* (q,p), i.e. in a different manifold that we can call $\tilde{\Sigma}$. The Hamiltonian for a single particle is defined as a Legendre transformation

$$H(q,p) = \sum_{i=1}^{d} p_i\dot{q}_i - L(q,\dot{q})\,, \tag{8}$$

where we have assumed that $\tilde{\Sigma}$ is $d-$dimensional. In a rather loose sense, in physics, the Hamiltonian is a measure of the energy of the system. In fact, in a lot of instances we can write $H = T + V$ without even bothering to go through the Lagrangian formalism. This is, in fact what a great part of physicists do, in particular those who work with quantum mechanics. In the more strict mathematical realm, the Hamiltonian is defined whenever time is a *cyclic* coordinate and the system is *closed*. Correspondingly, there is energy conservation and the Hamiltonian is identified as an *energy function*. In physics, we keep that intuition and interpretation even regardless of the previous assumptions[3].

Now, let us define two functions $A(q,p)$ and $B(q,p)$ living in phase space under the rule of a Hamiltonian $H(q,p)$. Then, we can define a *Poisson Bracket* between these two quantities as

$$\left[A,B\right]_{\text{P.B.}} \equiv \left\{A,B\right\}_{\text{P.B.}} = \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right). \tag{9}$$

From this definition we can observe that

$$\left\{q_i,q_j\right\}_{\text{P.B.}} = \left\{p_i,p_j\right\}_{\text{P.B.}} = 0\,, \tag{10}$$

$$\left\{q_i,p_j\right\}_{\text{P.B.}} = \delta_{ij}\,. \tag{11}$$

[3] We just manage to run away with it and then find the way to make everything consistent. For instance, in statistical mechanics we play with different ensembles not necessarily conserving energy. In quantum mechanics we play with non-unitary evolutions and extend the formalism accordingly.

This is known as the *canonical structure*. Also, recalling that $q = q(t)$ and $p = p(t)$, we see that

$$\frac{dA}{dt} = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial q_i}{\partial t} - \frac{\partial A}{\partial p_i} \frac{\partial p_i}{\partial t} \right) = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = \left\{ A, H \right\}_{\text{P.B.}} . \tag{12}$$

It immediately follows that if $\left\{ A, H \right\}_{\text{P.B.}} = 0$, then $A(q, p)$ is a *conserved quantity* or a *symmetry* of the system.

At this point, life is good! We have a good control over our system. In other words we can:

- Study symmetries at a formal level and check which quantities are conserved. We can also track symmetry breaking when playing with the free parameters of our theory.
- We can find the spectrum of our system, i.e. ground states and excitations by studying the Hamiltonian. For instance, in a many-body system, we can study phases of matter and their properties. Or in an error correction context we can find the codewords and errors of the code.
- We can study the propagation of the system, e.g. kinematics and dynamics, through Euler-Lagrange equations of motion and their solutions.
- We can study the statistical mechanical properties and thermodynamics of the system by computing the partition function as $Z(\beta) = \text{Tr}\left(e^{-\beta H}\right)$ for $\beta > 0$. Typically $\beta = 1/(k_B T)$ but it can be a different parameter, for instance (imaginary) time.

My first quantisation

We can also *quantise* our system through a *canonical quantisation* protocol based on making the replacements:

$$q_i, p_i \longrightarrow \hat{q}_i, \hat{p}_i \qquad \text{and} \qquad \left\{ q_i, p_j \right\}_{\text{P.B.}} = \delta_{ij} \longrightarrow \left[\hat{q}_i, \hat{p}_j \right] = i\hbar \delta_{ij} . \tag{13}$$

Here, notation $A \to \hat{A}$ denotes A, being a variable, becoming an operator-valued. Thus, generalised coordinates and their canonically conjugate momenta are replaced by non-commuting operators. This is, Poisson brackets are replaced by *commutators* defined as $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$. We can also define a "quantum" version of Eq. (12) which we write as

$$i\hbar \frac{d\hat{A}}{dt} = \left[\hat{A}, \hat{H} \right] , \tag{14}$$

which is known as *Heisenberg's equation of motion* and describes the evolution over time of an observable $\hat{A}$ in the so-called Heisenberg picture of Quantum Mechanics.

2 Index Notation is Superior

So far we have been dealing with generalised coordinates $\{q_i\}$. It is natural in physics to think of the concrete scenario in which these parameters are *position coordinates*, meaning $q_i = x_i$. We organise these coordinates in vectors $\mathbf{x} = (x^1, x^2, \ldots, x^d)$ for more compact notation. Then, for a given problem, we make a choice of coordinates, e.g. in 3-dimensional Euclidean space we use cartesian $\mathbf{x} = (x, y, z)$, spherical $\mathbf{x} = (r, \theta, \varphi)$ or cylindrical $\mathbf{x} = (\rho, \varphi, z)$ coordinates amongst others.

Furthermore, it is often the case that *time* is considered as an additional coordinate that we can add to the previous position vector. We then move from a *space manifold* description to a *spacetime manifold*. Notation generalises accordingly as

$$x \equiv x^{\mu} = (ct, \mathbf{x}) = (x^0, x^i) = (x^0, x^1, x^2, \dots, x^d)\,, \tag{15}$$

where c is a (universal) constant, namely the speed of light, so that $x^0 = ct$ has units of `Length`. In certain circles, this new vector x^{μ} is known as the $D-$position. For the rest of the lectures we will suppress this constant, i.e. work in units such that $c = 1$. We observe that the spacetime vector x has now $D = d + 1$ spacetime components for d space(-only) components. Therefore, we will refer to systems living in $(d+1)D$ spacetime dimensions, d of which are space-like and 1 is time-like.

The above notion can be extended for most physically relevant quantities. For instance, we can define the $D-$momentum

$$p \equiv p^{\mu} = \left(\frac{1}{c}E, \mathbf{p}\right) = (p^0, p^i) = (p^0, p^1, p^2, \dots, p^d)\,, \tag{16}$$

where E is the energy of the system. We can also define the spacetime (partial) derivative

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^i}\right) = (\partial_0, \partial_i)\,; \tag{17}$$

or a $D-$vector function of position

$$A^{\mu}(x^{\alpha}) = \Big(A^0(x^{\alpha}), A^i(x^{\alpha})\Big) = \Big(A^0(x^0, x^1, \dots, x^d), A^i(x^0, x^1, \dots, x^d)\Big) \tag{18}$$

as well as plenty of other objects. Now, there are several important remarks in using index or tensor notation. The first is identifying *greek indices* as *spacetime labels* $\mu, \nu, \lambda, \alpha, \dots \in [0, \dots, d]$, while *latin indices* are used for *space labelling* $i, j, k, l, \dots \in [1, \dots, d]$. Another important aspect is observing that sometimes quantities have "up" indices (e.g. x^{α}), known as *contravariant* quantities (or simply *up*); and some have "down" indices (e.g. x_{α}), known as *covariant* (or simply *down*). Whether indices are up or down matters! In fact, there are rules to *raise* or *lower indices*. We must *contract*[4] indices with the *metric* $g_{\mu\nu}(x)$. The metric is a rank-2 symmetric tensor with real eigenvalues — *don't ever quote me on this, but it is essentially a real square matrix* — that encodes the geometry of a spacetime manifold. It is defined by

$$ds^2 = g_{\mu\nu}(x)\, dx^{\mu} dx^{\nu}\,, \tag{19}$$

where ds is the *line element* or infinitesimal displacement vector in a given space. The metric also satisfies property $g_{\mu\nu}g^{\nu\lambda} = \delta^{\lambda}_{\mu}$, which essentially tells you how to take the inverse of the metric.

It is usually implicitly assumed, when working with vectors, that we are in *flat space*, this is $d-$dimensional *Euclidean space* Σ, which is locally $\Sigma \simeq \mathbb{R}^d$. The metric corresponding to

[4]Contracting indices means that we match up-down indices and we sum over them, so that we obtain a scalar. For instance, given quantities A^{μ} and B^{ν}, we might contract indices of A with B by taking $A^{\mu}B_{\mu} = A_{\mu}B^{\mu} \equiv \sum_{\mu=0}^{d} A_{\mu}B^{\mu} = A_0B^0 + A_1B^1 + \dots A_dB^d$. It is a convention, known as *Einstein's sum convention*, to not write the sum, but implicitly assume it every time we contract indices.

such a manifold is rather simple, it is $g_{ij} = \delta_{ij} = \text{diag}(+1,+1,\ldots,+1) = \mathbb{I}_{d\times d}$. Lowering (or raising) indices in this particular case amounts to performing

$$x^i = g^{ij}x_j = \delta^{ij}x_j = x_i \,, \tag{20}$$

which is a triviality. So, when dealing with vectors, we usually do not care about whether indices are up or down. However, when the space is **not** flat (or Euclidean) and/or we are dealing with space**times** , we **do** care! Flat spacetime is not, strictly speaking, Euclidean space[5]. Instead, it is called Minkowski spacetime (manifold) and is locally $M^D \simeq \mathbb{R}^{1,d}$ as it incorporates time. It is parametrised by the Minkowski metric denoted as $\eta_{\mu\nu}$, which corresponds to the case

$$g_{\mu\nu} = \eta_{\mu\nu} = (-1,\delta_{ij}) = \text{diag}(-1,+1,+1,\ldots,+1)\,. \tag{21}$$

Hence, we see that

$$x^\mu = \eta^{\mu\nu}x_\nu = (\eta^{00}x_0,\eta^{11}x_1,\eta^{22}x_2,\ldots,\eta^{dd}x_d) = (-x_0,+x_1,\ldots,+x_d) = (-x_0,x_i)\,. \tag{22}$$

as non-diagonal elements of the Minkowski metric vanish. This, of course, is a rule that has to be followed not only for positions x but, for any vector in spacetime. We can check explicitly, for instance, that $p^\mu = -i\hbar\partial^\mu$. Notice also that the general notions of scalar products, vector products, gradients, curls, etc. that we learn in undergraduate calculus now generalise. In particular it is useful to make use of the Levi-Civita symbol $\epsilon^{\mu\nu\lambda\alpha\ldots}$ which takes the value +1 whenever indices are ordered as

$$\epsilon^{0123\ldots} = -\epsilon_{0123\ldots} = +1 \tag{23}$$

or there is an even permutation of indices. If indices are permuted an odd number of times, it acquires the value $\epsilon^{1023\ldots} = -1$ and, when there are repeated indices it has value $\epsilon^{0112\ldots} = 0$.

Why bother ?

At this point we might wonder why should we even care about more sophisticated notation if we do not necessarily work with fancy curved spacetimes. There are several reasons why this is still a very useful notation:

- It is both extremely compact and general.
- It is natural once you learn the basic rules.
- It allows you to incorporate time naturally in the formalism without giving it any special treatment.
- It is much more intuitive for observing geometric, topological and dimensionality structures.

Let us write Maxwell's equations in the presence of sources in index notation. This is just

$$\partial_\nu F^{\mu\nu} = J^\mu \quad \text{and} \quad \partial_\nu \tilde{F}^{\mu\nu} = 0\,. \tag{24}$$

[5] You can also say it is a Euclidean space with Lorentzian signature, which can be turned into Euclidean signature by means of a Wick rotation $t = -i\tau$, essentially turning time into a space-like coordinate, thus Euclidianising Minkowski's space. This also allows to connect statistical mechanics with quantum mechanics. For now, though, we will not do imaginary-time magic and will stay in the Lorentzian signature.

Doesn't it look much more compact and graceful than the usual vector identities that come to mind when we think about Maxwell's equations? The attentive reader can start to see some resemblance between the two equations in Eq. (24). We will explore this in the future.

With regards to the latter bullet point of the list, even in plain Euclidean space, try to compute the following quantity $\mathbf{\nabla} \times \mathbf{B}$ in $2d$, $3d$ and $4d$, using conventional vector notation or index notation. We see that in index notation we can make use of contraction of indices to get intuition on the type of object resulting from such a product. More explicitly,

$$d = 2 : \quad \epsilon^{ij}\partial_i B_j = \phi \quad \text{(scalar quantity)}\,, \tag{25}$$

$$d = 3 : \quad \epsilon^{ijk}\partial_j B_k = \phi^i \quad \text{(vector quantity)}\,, \tag{26}$$

$$d = 4 : \quad \epsilon^{ijkl}\partial_k B_l = \phi^{ij} \quad \text{(tensor quantity)}\,. \tag{27}$$

In other words, dimensionality is linked to the number of indices. Also, vector calculus becomes a game of matching and counting indices, so complicated vector identities arise naturally using this language.

The link with gravity as devised by Einstein here is also straightforward. Einstein's equations for General Relativity are nothing but Euler-Lagrange equations where the generalised coordinate(s) $q_i(t)$ are replaced by the spacetime metric $g_{\mu\nu}(x)$. In other words, solving the (Einstein's) equations gives the explicit form of the metric $g_{\mu\nu}(x)$ as solutions. In the trivial, flat case, the solution is the Minkowski metric. A slightly non-trivial spherically-symmetric case is that of a static black hole, known as the Schwarzchild metric solution. The interested reader is referred to Appendix 4.2 for more details.

3 What is a Field?

A field is a physical quantity that has a value at each point of some parametric space. Typically this parametric space is spacetime. We can classify fields according to different criteria, most of them partially overlapping, but that provide intuition about the type of object we are dealing with. For instance, fields can be:

- Scalar, vectorial, spinorial or tensorial quantities amongst others.
- Elements of a group G, being this group the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, angles or phases S^1, Integers $\mathbb{Z}$ modulo n, etc.
- Classical or quantum in that they are functions or operators.
- ... Other criteria.

In a more mathematical stance, we say that fields are maps $\phi : \Sigma \longrightarrow \tilde{\Sigma}$ so that the field ϕ has an input variable, typically x, that lives in Σ, and the image, typically $\phi(x)$, lives on the space $\tilde{\Sigma}$. This is a very natural and yet general notion that we will make continuous use of, so it is worth discussing basic examples that allow us identify, classify and manipulate fields in physics. For instance:

- The position $\mathbf{x}(t) \in \mathbb{R}^d$ is a real, d—dimensional classical, vector field on a (one-dimensional) time manifold.
- The wavefunction $\psi(x^\alpha) = \psi(t,\mathbf{x}) \in \mathbb{C}$ is a classical, complex-valued, scalar field on a D—dimensional spacetime manifold.
- The temperature $T(\mathbf{x}) \in \mathbb{R}^+$ is a classical, positive-real, scalar field on a d—dimensional space manifold.
- The metric $g_{\mu\nu}(x) \in \mathbb{R}^{D\times D}$ is a classical, real-valued, tensorial field on a D—dimensional (pseudo-Riemannian) spacetime manifold.
- The electric field $\mathbf{E}(x^\alpha) \in \mathbb{R}^d$ (or $\hat{\mathbf{E}}(x^\alpha) \in \mathcal{F}_+$) is a classical (or quantum), real-valued (or Fock-bosonic-valued), d—dimensional vector field on a D—dimensional spacetime manifold.
- People in "quantum gravity" look for a well-defined object $\hat{g}_{\mu\nu}(x)$ that is the quantum version of the spacetime metric (field).

A particularly interesting object, which will be the subject of this course, is the field

$$A^\mu(x^\alpha) = A^\mu(t,\mathbf{x}) = (A^0(x), A^i(x)) = (\varphi, \mathbf{A}) \in \mathbb{R}^D \ . \tag{28}$$

This is an example of an (Abelian) gauge field. For now, we see it is a classical, real-valued, D—dimensional vector field on a D—dimensional spacetime manifold. In $(3+1)D$ and in the context of electromagnetism, we identify A^μ as the 4—vector potential, φ is known as the scalar potential and $\mathbf{A}$ is known as the magnetic vector potential. For us, $A^\mu(x)$ will just be called a classical gauge field.

A word of Caution: Electric and Magnetic fields don't exist!

In these notes we will want to detach from the idea of vector potentials being defined in terms of electric and magnetic fields. We will work under the assumption that electric and magnetic fields are not defined (until we say so). In other words, we will define them in terms of A^μ and not the other way around. We will see this is much more powerful and natural.

To some extent this is analogous to stating that the wavefunction $\psi(t,\mathbf{x})$ is the fundamental building block for quantum mechanics and not the *amplitudes* or *probabilities*, which are the actual measurable (physical) outcomes. So bear with me, and assume electric and magnetic fields do not exist (yet)!

3.1 Field Strength

Whenever we have fields, we can define a *field strength* as the *magnitude* of a field. For a vector field, this is typically a generalised sense of the "modulus" of a given quantity or, more informally, "how long is the arrow of the vector". Examples of this are the wind speed — being, let's say 30km/h — for a given x in spacetime. Another example is the (normalised) probability $|\psi(x)|^2 = 0.3$ (for instance).

In the case of a gauge field, computing its field strength can be found by finding the *curvature 2-form* of an (Abelian gauge) *connection 1-form*. This is, defining the object $A \equiv A_\mu dx^\mu$, we compute the object $F = dA = F_{\mu\nu}\, dx^\mu \wedge dx^\nu$. We will need a bit more advanced maths to see where this all comes from[6], we will see this in future sessions. The point here is that we can construct an object called the *curvature*[7] or *field strength* of the gauge field which has the form

$$F^{\mu\nu}(x) \equiv \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\,. \tag{29}$$

The construction of this object is purely based on geometry, without any physical input. It just so happens that this object has the same form as the well-known electromagnetic (field-strength) tensor. It is only now that we can observe that the tensor is antisymmetric, with $F^{00} = \partial^0 A^0 - \partial^0 A^0 = 0$ and $F^{ii} = \partial^i A^i - \partial^i A^i = 0$, but $F^{0i} = \partial^0 A^i - \partial^i A^0 \neq 0$ and $F^{ij} = \partial^i A^j - \partial^j A^i \neq 0$ for $i \neq j$. Hence, we **define** the electric and magnetic fields as non-zero components of the field strength, namely

$$F^{0i} \equiv E^i = \mathbf{E} \tag{30}$$

and

$$F^{ij} \equiv \epsilon^{ijk} B_k\,, \qquad \text{so} \qquad B^i = \frac{1}{2}\epsilon^{ijk} F_{jk} \equiv \mathbf{B} \tag{31}$$

We can easily express these definitions in components as

$$\mathbf{E} \equiv F^{0i} = \partial^0 A^i - \partial^i A^0 = -\partial_0 A^i - \partial_i A^0 = -\frac{\partial}{\partial t}\mathbf{A}(x) - \boldsymbol{\nabla}\varphi(x) \tag{32}$$

and

$$\mathbf{B} \equiv \frac{1}{2}\epsilon^{ijk} F_{jk} = \frac{1}{2}\Big(\epsilon^{ijk}(\partial_j A_k - \partial_k A_j)\Big) = \frac{1}{2}\Big(2\epsilon^{ijk}(\partial_j A_k)\Big) = \boldsymbol{\nabla} \times \mathbf{A}(x)\,. \tag{33}$$

We have defined electric and magnetic fields as certain components of the field strength of an underlying gauge field A^μ.

Gauge invariance

A crucial property of the field strength $F_{\mu\nu}$ is that it is *gauge invariant*, i.e. unchanged under local transformations, also known as *gauge transformations*. Gauge transformations are parametrised by a unitary and invertible matrix $\mathcal{U}(x)$, which is much more restrictive than usual global symmetries merely acting on the system as a whole. For an Abelian *gauge group* U(1), the form of the matrix is $\mathcal{U}(x) = \exp[i\xi(x)]$, where ξ is a real function. Thus, the gauge field locally transforms according to

$$A_\mu(x) \longrightarrow A'_\mu(x) = \mathcal{U}(x) A_\mu(x) \mathcal{U}^{-1}(x) + \frac{i}{g}\,\mathcal{U}(x)\,\partial_\mu \mathcal{U}^{-1}(x)\,, \tag{34}$$

where $\mathcal{U}^{-1}(x)\mathcal{U}(x) = \hat{\mathbb{1}}$ and g is the coupling constant, typically the electric charge. This expression reduces to

$$A'_\mu(x) = A_\mu(x) + g^{-1}\,\partial_\mu\,\xi(x) \tag{35}$$

for this particular Abelian case. Notice A_μ and A'_μ correspond to different configurations of the same physical system, the same equivalence class, meaning there is an intrinsic *gauge redundancy* or *freedom* to choose any of them. For practical computation and manipulation it is often necessary to choose one of these configurations to work with.

[6]Here this new notation is called *differential form* notation or $p-$form language, which is a slightly more modern and sophisticated evolution of index notation. It is used in *exterior calculus* and it becomes very geometrically intuitive. The object d is an *exterior derivative*, while $\wedge$ constitutes the so-called *wedge product*.

[7]Yes, it is the same curvature as in curved spacetime from Einstein or curvature as in that of a sphere.

That is called *gauge fixing* or choosing a gauge. Once again, it is important to stress that choosing a gauge, like choosing a preferred system of coordinates, cannot affect the Physics, only make it more or less obscure.

Exercise. Use Eq.(35) to show that the field strength is gauge invariant, i.e. $F_{\mu\nu} = F'_{\mu\nu}$.

3.2 Constructing an Action

A physical theory can be defined as an adequate action principle representing the system under study. This is, constructing a Lagrangian density $\mathcal{L}$ that is a function of relevant fields and that, when integrated over the appropriate variables, returns a number. If the action has as inputs $S[A_\mu, \partial_\nu A_\mu] = \int d^D x\, \mathcal{L}$ it is known as a *gauge (field) theory*. The action must be gauge invariant so that it can return a single number as output. Otherwise, the action would depend on an arbitrary choice of the (local) gauge function $\xi(x)$, which is not physical[8]. In addition, we want all the possible indices to be contracted as the Lagrangian density is a scalar quantity. Finally, we want this action principle to be as simple as possible, meaning we want the minimal number of fields and derivatives involved[9].

From the previous discussion we have found a gauge-invariant and physically relevant quantity, the field strength. So constructing an action based on the field strength ensures gauge invariance. A possible choice for a Lagrangian density could be F^{μ}_{μ}, but this is identically zero by construction. The next-order options are $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\alpha}F_{\mu\nu}F_{\lambda\alpha}$, both yielding a scalar as all indices are contracted. The former construction corresponds to Maxwell's theory of electromagnetism, while the latter is known as the *theta term* (or the related concept of *axion electrodynamics*). We will study the effect of the theta term in future sessions, as well as another interesting term known as the Chern-Simons term, namely $\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}$. Regarding the former construction, adding a constant in front, yields classical electromagnetism as the simplest *consistent* theory for an Abelian gauge field

$$S_M = \int d^D x\, \mathcal{L}_M = -\frac{1}{4}\int dt\, d^d\mathbf{x}\, F_{\mu\nu}F^{\mu\nu} \tag{36}$$

Exercise. From Maxwell's action (36), find the inhomogeneous Maxwell's equations in vacuum $\partial_\nu F^{\mu\nu} = 0$ and write them in vector notation. Find the homogeneous Maxwell's equations from $\partial_\nu \tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\lambda\alpha}\partial_\nu F_{\lambda\alpha} = 0$.

Can you add the presence of an electric source $J^\mu = (J^0, J^i) = (\rho, \mathbf{J})$ to the inhomogeneous equations, where ρ is a charge density and $\mathbf{J}$ is an electric current density? Can you add now the presence of a magnetic source $\tilde{J}^\mu = (\tilde{J}^0, \tilde{J}^i) = (\tilde{\rho}, \tilde{\mathbf{J}})$ to the homogeneous equations?

[8] This is as if the physical content of a theory was dependent on the system of coordinates used. We might have a preferred system of coordinates, but the physical properties cannot depend on the choice of coordinates. Analogously, a theory cannot depend on the preferred choice of gauge, i.e. of the function $\xi(x)$.

[9] There are Renormalisation Group (RG) arguments that make this qualitative remark more concrete and formal, but they are not simple. The general argument is that simpler terms tend to dominate while nonlinearities or high many-body terms tend to be non-dominant.

4 Bonus*: Canonical Schrödinger and Maxwell

4.1 Canonical Quantisation of the Schrödinger Field

From a field theoretic perspective, the Schrödinger field is a *non-relativistic*, *complex*, *scalar* field. In the following, we will also assume it is *bosonic*, although a fermionic treatment would be analogous. The starting point of the canonical treatment consists on defining an action

$$S = \int dt\, d^d\mathbf{x}\, \mathcal{L}\left[\Psi, \Psi^\dagger\right] = \int dt\, d^d\mathbf{x}\, \Psi^\dagger(t,\mathbf{x})\left[\hat{E} - \frac{\hat{\mathbf{p}}^2}{2m} - \hat{V}(\mathbf{x})\right]\Psi(t,\mathbf{x}) \tag{37}$$

which we will extremise. According to first quantisation, the energy operator is $\hat{E} = i\hbar\partial_t$, while momentum is given by $\hat{\mathbf{p}} = -i\hbar\boldsymbol{\nabla}$. Extremisation of the action with respect to Ψ yields

$$\frac{\delta S}{\delta\Psi} = \frac{\delta\mathcal{L}}{\delta\Psi(x)} - \partial_\mu\, \Pi^\mu_\Psi(x) = 0\,, \tag{38}$$

where we have used the notation $x \equiv x^\mu = (t,\mathbf{x})$ and defined the canonically conjugate momentum to Ψ as

$$\Pi^\mu_\Psi(x) \equiv \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\Psi(x)\right)}\,. \tag{39}$$

After integrating by parts, with appropriate boundary conditions, we find the Lagrangian density

$$\mathcal{L} = i\hbar\Psi^\dagger(x)\partial_t\Psi(x) - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\Psi^\dagger(x)\cdot\boldsymbol{\nabla}\Psi(x) - V(\mathbf{x})\Psi^\dagger(x)\Psi(x) \tag{40}$$

The Hamiltonian density can then be found as $\mathcal{H} = \Pi^t_\Psi(x)\partial_t\Psi(x) - \mathcal{L}$, yielding the field theoretical Hamiltonian

$$H = \int d^d\mathbf{x}\,\mathcal{H} = \int d^d\mathbf{x}\,\frac{\hbar^2}{2m}\boldsymbol{\nabla}\Psi^\dagger(x)\cdot\boldsymbol{\nabla}\Psi(x) + V(\mathbf{x})\Psi^\dagger(x)\Psi(x) = \frac{1}{i\hbar}\int d^d\mathbf{x}\,\Pi^t_\Psi(x)\,\mathcal{H}_{QM}\,\Psi(x) \tag{41}$$

where $\mathcal{H}_{QM}$ denotes the usual quantum-mechanical Hamiltonian density. Canonical quantisation follows from postulating that a given field and its canonically conjugate momentum satisfy

$$\left[\hat{\Psi}(x), \hat{\Psi}(x')\right]_\mp = \left[\hat{\Pi}^t_\Psi(x), \hat{\Pi}^t_\Psi(x')\right]_\mp = 0 \tag{42}$$

$$\left[\hat{\Psi}(x), \hat{\Pi}^t_\Psi(x')\right]_\mp = i\hbar\,\delta^{(D)}\!\left(x - x'\right) \tag{43}$$

4.2 Canonical Quantisation of a Gauge Field

> *Exercise.* From the Maxwell action in Eq.(36), find the Lagrangian and Hamiltonian densities expressed in terms of the electric **E** and magnetic **B** fields. Construct the canonical commutation relations and try to quantise electromagnetism. What happens?

Acknowledgements

I want to thank the lads from SciPost for creating and publicly sharing this beautiful template, which I borrowed and adapted for this set of notes. You can also find it and enjoy it at https://scipost.org/media/uploads/templates/SciPostPhysLectNotes/2023/SciPostPhysLectNotes_LaTeX_Template.tgz

A Some comments on curved spacetimes

Getting familiar with the metric

The metric tensor $g^{\mu\nu}(x)$ is not an exotic object, it is just a reformulation of things that we already know. Let us review a couple of examples.

Flat 3d Euclidean Space. Real space is parametrised as

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \,. \tag{A.1}$$

This is, the metric is diagonal with components $g^{11} = +1$, $g^{22} = +1$ and $g^{33} = +1$. In summary, we find $g_{ij} = \delta_{ij}$

The 2-sphere. The usual way to parametrise points in a regular sphere is using spherical angular coordinates, so $x^i = (\theta, \varphi)$. The line element is

$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 \,. \tag{A.2}$$

Thus, the metric is diagonal with only two non-zero elements $g_{11} = +1$ and $g_{22} = \sin^2\theta$.

Schwarschild Black Hole and Rindler Observer. The Schwarzschild metric is as spherically symmetric solution of Einstein's equations for an empty universe ($T^{\mu\nu} = 0$) and has the form

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2\,d\Omega_{D-1}^2 \,, \tag{A.3}$$

where $f(r) = 1 - M\frac{w_d}{r^{d-2}}$, and

$$w_d = \frac{16\pi G_N}{(d-1)\mathrm{Vol}(\mathbf{S}^{d-1})} \tag{A.4}$$

is essentially a volume factor. For instance, for a one-dimensional $\mathrm{Vol}(\mathbf{S}^1(R)) = 2\pi R$, two-dimensional $\mathrm{Vol}(\mathbf{S}^2(R)) = 4\pi R^2$ or, more generally, $(d-1)$-dimensional $\mathrm{Vol}(\mathbf{S}^{d-1}(R)) = R^{d-1}\mathrm{Vol}(\mathbf{S}^{d-1})$ sphere with $\mathrm{Vol}(\mathbf{S}^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$.

For $D = 3+1$ this solution reduces to $f(r) = 1 - \frac{2MG_N}{r}$, and the Schwarzschild radius is defined as $r_s = 2MG_N$. More generally, we can consider that $f(r)$ has some root at $\tilde{r}$ so that $f(r = \tilde{r}) = 0$ and the metric blows up. This coordinate singularity defines an *(event) horizon*. We can study the near-horizon limit of the metric by considering $r = \tilde{r} + \epsilon$ for $\epsilon \ll 1$. We can then Taylor expand to first order

$$f(r)|_{r\to\tilde{r}} = f(r)|_{r=\tilde{r}} + f'(r)|_{r=\tilde{r}}(r-\tilde{r}) + \ldots \,, \tag{A.5}$$

but the zeroth order of the expansion vanishes identically by definition. Hence the near-horizon metric becomes

$$ds^2 \approx ds_{nh}^2 = -f'(\tilde{r})(r-\tilde{r})\,dt^2 + \frac{dr^2}{f'(\tilde{r})(r-\tilde{r})} + \tilde{r}^2\,d\Omega_{D-1}^2 \,. \tag{A.6}$$

From here we can define the *surface gravity* as $\kappa \equiv f'(\tilde{r})/2$, substitute it and apply the change of variables

$$d\rho^2 = \frac{dr^2}{2\kappa\,(r-\tilde{r})} \tag{A.7}$$

from which we can take the square root and integrate to find

$$\rho = \frac{1}{\kappa}\sqrt{2\kappa\,(r-\tilde{r})}\,. \tag{A.8}$$

Thus, the near-horizon metric becomes

$$ds^2_{nh} = -(\kappa\rho)^2\,dt^2 + d\rho^2 + \tilde{r}^2\,d\Omega^2_{D-1}\,. \tag{A.9}$$

These are the so called Rindler coordinates, which describe a Minkowski spacetime frome the point of view of a (Rindler) observer moving at an acceleraction κ which, in this case, corresponds to the surface gravity of what can be a black hole. In fact, one can transform back into Minkowski space via the change

$$T = \frac{1}{\kappa}\,\rho\,\sinh(\kappa t) \tag{A.10}$$

$$R = \frac{1}{\kappa}\,\rho\,\cosh(\kappa t) \tag{A.11}$$

so that the metric is re-expressed as

$$ds^2_{nh} = -dT^2 + dR^2 + \tilde{r}^2\,d\Omega^2_{D-1}\,. \tag{A.12}$$

Hence, we have seen that Rindler is the near-horizon limit of Schwarzschild.

FLRW Universe. For an isotropic, homogeneous and expanding universe we have

$$ds^2 = -dt^2 + a(t)^2\left(\frac{1}{1-kr^2}\,dr^2 + r^2 d\Omega^2_2\right), \tag{A.13}$$

where k is a curvature parameter and $a(t)$ is the *expansion parameter* of the Universe.

Gravitational waves. We can write the metric for gravitational waves as a small disturbance $\tilde{g}^{\mu\nu}$ over a background metric $h^{\mu\nu}$

$$g_{\mu\nu} = h_{\mu\nu} + \tilde{g}_{\mu\nu}\,. \tag{A.14}$$

General Relativity in a Nutshell

This box is a lightning review of General Relativity. Einstein observed:

- *Gravity is Spacetime Geometry.*
- *Matter Sources Gravity.*

Based on the above, one can propose the principles:

- *Principle of General Relativity:* All laws of Physics take the same forms in any coordinate system.
- *Principle of Equivalence:* There exists a coordinate system in which the effect of a gravitational field vanishes locally, i.e. an observer in free fall does not "feel" gravity. In other words, an *inertial* mass is indistinguishable from a *gravitational* mass. Hence, *gravity* is locally indistinguishable from *acceleration*.

General Relativity describes the dynamics of spacetime geometry encoded in a metric tensor $g_{\mu\nu}(x)$. It can be formulated from an action principle of the form

$$S_{\mathrm{EH}} = \frac{1}{\kappa_D} \int d^D x \, \sqrt{|g|} \left(R + \mathcal{L}_{\mathrm{matter}} \right), \tag{A.15}$$

known as the Einstein-Hilbert action in the presence of matter. The constant $\kappa_D = 16\pi G_N^D = 2\, m_p^{-2}$ is introduced to recover the Newtonian limit, where G_N^D is the Newton's constant in the corresponding dimension and m_p is the reduced Planck mass in natural units. The Euler-Lagrange equations of motion for the metric are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\kappa_D}{2} T_{\mu\nu} \tag{A.16}$$

known as the Einstein's field equations, which relate "curvature" on the l.h.s. to "energy" on the r.h.s. of the equation. The Ricci curvature is given by

$$R_{\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\lambda\nu} - \Gamma^\lambda_{\mu\nu} \Gamma^\sigma_{\lambda\sigma} \tag{A.17}$$

where the Christoffel connections are defined as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \tag{A.18}$$

The Ricci scalar is simply $R \equiv R_{\mu\nu}\, g^{\mu\nu}$. On the other hand, the symmetric (or Belinfante) stress-energy-momentum tensor is

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{|g|}} \frac{\delta \mathcal{S}_{\mathrm{matter}}}{\delta g^{\mu\nu}} . \tag{A.19}$$

A free falling particle follows a curve known as a *geodesic*. The geodesic equation is

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{dt} \frac{dx^\lambda}{dt} = 0 . \tag{A.20}$$

Newton from Einstein

The Møller-Rosenfeld prescription for semiclassical gravity considers Einsteinian gravity coupled to quantum matter. Due to the incompatibility of coupling classical spacetime to operator-valued stress-energy-momentum tensor, one works with expectation values, i.e. $\langle \hat{T}_{\mu\nu} \rangle \equiv \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle$. This averaging, of course, neglects quantum fluctuations from matter. The Einstein field equations become (we explicitly write c for convenience)

$$G_{\mu\nu} = \frac{8\pi G_{\rm N}}{c^4} \langle \hat{T}_{\mu\nu} \rangle \ . \tag{A.21}$$

Now we will consider both a non-relativistic and a Newtonian limits. This translates into the following assumptions:

1. Weak gravitational field. Meaning

$$\frac{G_{\rm N}\, m}{r c^2} \ll 1 \ . \tag{A.22}$$

   In this limit, this allows for the expansion of the metric as nearly flat, in essence

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon\, h_{\mu\nu} \tag{A.23}$$

   for $\epsilon \ll 1$.

2. Source of gravitational field is mostly due to mass density.

$$\frac{|T^{ij}|}{T^{00}} = \frac{|T^{ij}|}{\rho_m c^2} \ll 1 \ . \tag{A.24}$$

3. Stress-energy-momentum (e.g. mass) sources move slowly.

$$v \ll c \ . \tag{A.25}$$

   This implies that time derivatives can be dropped.

Armed with this conditions we can see that for $\epsilon \ll 1$ and $|v^i| \ll 1$ implies $\frac{{\rm d}t}{{\rm d}\tau} \approx 1$. Then, the geodesic equation

$$\frac{{\rm d}^2 x^\mu}{{\rm d}\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{{\rm d}x^\nu}{{\rm d}\tau} \frac{{\rm d}x^\lambda}{{\rm d}\tau} = 0 \tag{A.26}$$

for $\mu = i$ and $c = 1$ reduces to

$$\frac{{\rm d}^2 x^i}{{\rm d}t^2} \approx \frac{{\rm d}^2 x^i}{{\rm d}\tau^2} = -\Gamma^i_{\nu\lambda} \frac{{\rm d}x^\nu}{{\rm d}\tau} \frac{{\rm d}x^\lambda}{{\rm d}\tau} = -\Gamma^i_{00} = -\Gamma_{i\,00} = \frac{1}{2}\partial_i h_{00} - \partial_0 h_{i0} = \frac{1}{2}\partial_i h_{00} \equiv \partial_i \phi \ , \tag{A.27}$$

so we end up with Newton's equation $\ddot{\mathbf{x}} + \boldsymbol{\nabla}\phi = 0$, with identification (restoring c) of the Newtonian scalar potential as the time component of the metric $h_{00} \approx -\frac{2}{c^2}\phi$ and $g_{00} \equiv \eta_{00} + \epsilon\, h_{00} \approx -(1 + \frac{2\phi}{c^2})$. This allows us now to compute

$$G_{00} = R_{00} = R^i_{0i0} + R^0_{000} = R^i_{0i0} = -\frac{\partial^2 h_{00}}{\partial x^i\, \partial x^i} = \frac{2}{c^2}\, \boldsymbol{\nabla}^2 \phi \tag{A.28}$$

$$= \frac{8\pi G_{\rm N}}{c^4} \langle \hat{T}_{00} \rangle = \frac{8\pi G_{\rm N}}{c^4} \langle \hat{\rho}_m c^2 \rangle = \frac{8\pi G_{\rm N}}{c^2} \langle \Psi |\, m\, | \Psi \rangle = \frac{8\pi G_{\rm N} m}{c^2} |\Psi|^2 \ . \tag{A.29}$$

In summary, the Einstein field equations linearise and reduce to

$$\boldsymbol{\nabla}^2 \phi\,(t, \mathbf{x}) = 4\pi m\, G_{\rm N}\, |\Psi(t, \mathbf{x})|^2 \tag{A.30}$$

in the non-relativistic Newtonian limit. This shows that the Gauss's law that was previously discussed can be found from conventional Møller-Rosenfeld semiclassical gravity.

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